IS THIS A COINCIDENCE? STUDENTS' UNDERSTANDING OF THE ROLE OF EXAMPLES IN PROVING OR REFUTING OF ALGEBRAIC STATEMENTS

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The task: Is this a coincidence?

A student took two fractions $\frac{1}{2}$ and $\frac{3}{4}$ and added the two numerators and the two denominators in the following way: $\frac{1+3}{2+4} = \frac{4}{6} = \frac{2}{3}$.

The student noticed that the resulting fraction $\frac{2}{3}$ is between the two original ones: $\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$.

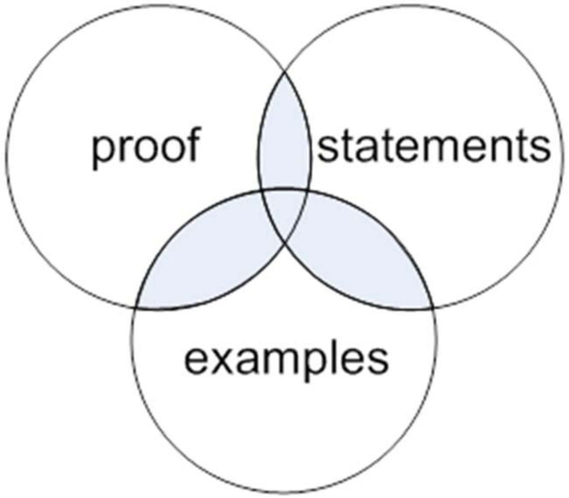
Is this a coincidence?

Features of the type of task: Is this a coincidence?

- The task deals with **familiar** mathematical content, but the observed phenomenon/problem is **not familiar** to the students.
- An emphasis on evoking **uncertainty** and fostering an intellectual need to resolve it by proof or refutation.
- The question: “Is this a coincidence?" invites the students to **evaluate the generality** of the observed phenomenon.
- Successful completion of the task involves formulating a conjecture and either **proving or refuting** it by a counterexample.

Background and the focus of our research

- Understanding of the logical relations between examples and statements is critical for proving, yet not trivial.
- This kind of understanding is not sufficiently addressed in school curriculum. Especially not in algebra.



- **Research goal:** To characterize students' understanding of the roles of examples in determining the validity of mathematical statements.

We build on the vast body of literature on Proof and Proving and on the role of Examples in mathematical thinking and learning.

Mathematical framework

What does it mean to understand the status of examples in proving or refuting mathematical statements, and how can this understanding be captured and characterized?

- The framework maps four types of examples: confirming, contradicting, non-confirming and irrelevant to two types of statements: universal and existential.
- We focus on **universal** statements and **partially** exemplify the framework with a specific (false) universal statement:

$$\textit{Every } \underbrace{\textit{integer ending with 7}}_{\text{Domain } D} \; \underbrace{\textit{is a multiple of 7}}_{\text{Proposition } P(x)}$$

The Domain: The Proposition:	*D*: All integers ending with 7 *P(x)*: Is a multiple of 7			
Type of Statement	**Universal statement** $\forall x \in D, P(x)$ *All integers ending with 7 are multiples of 7.*		**Existential Statement** $\exists x \in D, P(x)$ *There exists an integer ending with 7 that is a multiple of 7.*	
Goal / **Type of Example**	**To prove**	**To disprove**	**To prove**	**To disprove**
Confirming $x \in D, P(x)$ $x=77$	**Insufficient**	Not applicable	**Sufficient**	Not applicable
Contradicting the universal statement **Non-confirming** the existential statement $x \in D, \neg P(x)$ $x=17$	Not applicable	**Sufficient**	Not applicable	**Insufficient**
Irrelevant $x \notin D, P(x)$ $x=70$	Not applicable	Not applicable	Not applicable	Not applicable
Irrelevant $x \notin D, \neg P(x)$ $x=71$	Not applicable	Not applicable	Not applicable	Not applicable

We define “understanding of the status of examples in determining the validity of mathematical statements” in operational terms as:

Becoming fluent with the logical inferences that can and cannot be made, based on the different types of examples, with respect to the given statement.

In a task of the type: Is it a coincidence? this understanding can be manifested in:

Type of Statement	Universal statement	
Type of Example / Goal	To prove	To disprove
Confirming	Acknowledging that confirming examples are insufficient for proving a conjecture	
Contradicting the universal statement		Providing a counterexample to refute a false conjecture (when the phenomenon is a coincidence)
Irrelevant		

Constructing relevant examples and correctly identifying their status.

Methodology

- **Participants:**
  - Six pairs of 10^{th} grade "top-level" students, from 2 Israeli schools; 7 girls, 5 boys.
- **Data collection:**
  - Six, task-based, semi structured interviews with each pair of students.
  - Video recordings, transcripts, students' written work.
- **Tasks:**
  - Six types of tasks were developed in accordance with the conceptual framework. One type of task: ***Is this a coincidence?***
  - Two parallel versions for each type: algebra and geometry.
- **Data analysis:**
  - Unit of analysis: one pair of students' interaction with one task.
  - Coding of actions and utterances according to the conceptual framework.

IOU – indicators of understanding NNR – non-normative responses

Findings

What can we learn from the way students used examples in the task "Is this a coincidence?"

about their understanding of the roles of examples in proving or refuting universal algebraic statements?

Students identified the task as dealing with universal conjecture. Constructed relevant examples and correctly identified their status.

Type of Statement	Universal statement ✓			
Goal	To prove		To dispr ve	
Type of Example	**IOU** indicators of understanding	**NNR** non-normative responses	**IOU** indicators of understanding	
Confirming ✓				
Contradicting ✓ the universal statement				
Irrelevant ✓				

All students **correctly identified** the task as dealing with universal implicit conjecture.

All students **correctly constructed** examples and **identified** their status. Students **negotiated which examples are relevant** and which are not.

Students used examples to define the domain of the mathematical phenomenon.

Any two fractions.

Any proper fractions.

What are the two fractions $\frac{1}{2}$ and $\frac{3}{4}$ an example of?

Fractions constructed of 4 consecutive numbers like 1, 2, 3, 4.

Fractions which have the same difference between the denominator and the numerators : 2-1 = 4-3.

Students acknowledged that confirming examples are insufficient for proving a conjecture.

Nurit and Limor considered only fractions that can be constructed from 4 consecutive natural numbers.

They used the representation: $\frac{a}{a+1}$, $\frac{a+2}{a+3}$.

Adding the numerators and the denominators: $\frac{a+(a+2)}{(a+1)+(a+3)} = \frac{a+1}{a+2}$

Nurit and Limor proved algebraically that: $\frac{a}{a+1} < \frac{a+1}{a+2} < \frac{a+2}{a+3}$.

When asked whether the phenomenon is a coincidence or not Nurit replied:

It is [a coincidence], *because by our tests and proof, it is true only for consecutive natural numbers. But at this time we don't have tools for checking this for non-consecutive numbers. Look, it can be true, but it can also be that with the non-consecutive numbers I will find 2000 examples that it is true, but* ***in the example 2001 it will be false, and then everything is false****.*

Students used systematically generated examples to "prove" a conjecture.

Neta and Ronit divided the domain of fractions into subdomains. They "proved" that the phenomenon is a general rule by examining one, randomly chosen example in each domain.

They wrote:

Both fractions smaller then 1 : $\frac{1+1}{4+2} = \frac{1}{3} \rightarrow \frac{1}{4} < \frac{1}{3} < \frac{1}{2}$ *true.*

One fraction <1, another fraction >1: $\frac{2+6}{3+3} = \frac{4}{3} \rightarrow \frac{2}{3} < \frac{4}{3} < \frac{6}{2}$ *true.*

Both fractions bigger then 1 : $\frac{5+6}{2+4} = \frac{11}{6} \rightarrow \frac{6}{4} < \frac{11}{6} < \frac{5}{2}$ *true.*

In order to prove that when you add two numerators and two denominators the resulting fraction is always between the original ones, you need to ***prove it with other cases*** *that will fit the rule. Like here we tried different ones and we proved it…that it is not a coincidence.*

What can we learn about students' understanding of the roles of confirming examples in proving universal algebraic statements?

Type of Statement	Universal statement			
Goal	To prove		To disprove	
Type of Example	**IOU** indicators of understanding	**NNR** non-normative responses	**IOU** indicators of understanding	**NNR** non-normative responses
Confirming	✓	✗		

Nurit and Limor showed explicit understanding that **confirming examples are insufficient for proving**.

Neta and Ronit **used** systematically generated **confirming examples to "prove"** the universal statement.

Students used counterexamples to refute a conjecture.

Ben and Keren used similar strategy of checking three cases: two fractions which are smaller than 1; one fraction which is smaller than 1 and another fraction which is greater than 1; (3) two fractions that are greater than 1.

However, they made a computational mistake in the 3rd case. They took $\frac{9}{8}$ and $\frac{4}{3}$ and got: $\frac{9+4}{8+3} = \frac{13}{12} \rightarrow \frac{4}{3} > \frac{9}{8} > \frac{13}{12}$

The resulting fraction is not between the two original ones. Ben and Keren interpreted this as a counterexample.

They explained: *We take two fractions greater than 1, we check and see that the resulting fraction is the smallest, and we say that the rule is not general. We refuted his [a hypothetical student's] statement.*

What can we learn about students' understanding of the role of contradicting example in refuting of universal algebraic statements?

Type of Statement	Universal statement			
Goal	To prove		To disprove	
Type of Example	IOU	NNR	IOU indicators of understanding	NNR non-normative responses
Confirming				
Contradicting the universal statement			✓	
Irrelevant				

Nurit and Limor explicity stated that one **counterexample** is sufficient to **refute** a conjecture.

Ben and Keren interpreted one of their examples as **counterexample** and used it to **refute** a conjecture.

Conclusions and contribution of the study

1. The **conceptual framework** proved to be useful in:

   - **Designing** tasks that both **asses** and **promote** the development of students' understanding of the roles of examples in proving.

   - **Characterizing** students' understanding of these roles.

2. The task ***"Is it a coincidence?"*** is a powerful trigger to **engage** students in: making, exploring, proving or refuting mathematical conjectures.

3. As a **diagnostic tool**, the task "Is it a coincidence?" helped to reveal aspects of students' understanding of the roles of examples in proving or refuting (algebraic) conjectures.

4. There is a need for careful design and scaffolding of **instructional support** for students.

Thank you!