Counter Examples in Mathematics: Generation Processes and Modes of their Use

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The present study examines the learning processes and student understanding related to the concept of counterexamples: the ways in which students understand and use the concept of counterexamples. In particular, the study concentrates on development of understanding of the role counterexamples play in refuting false conjectures, the tendency to use counterexamples in checking and evaluating conjectures and the ways students construct and use them.

In the research literature there is a lot of empirical evidence that indicates that students have difficulties with understanding the concept of mathematical deduction and that they tend to relay too much on inductive inference.

Counterexamples have a great importance as means for refuting a false conjectures and the use of counterexamples might help in developing more cautious use of inductive inference.

Despite the seeming simplicity of counterexamples, empirical studies showed that students sometimes posses wrong conceptions associated with counterexamples, their generation and usage. For instance, many students don't find a single counterexample as a sufficient proof of a fallacy of a mathematical statement and tend to reject or treat counterexamples as exceptions.

Other difficulties associated with the concept of counterexamples are the tendency to limit the domain of examples to be checked for evaluating an algebraic statement to integers and also some logical aspects of counterexample use like inability to distinguish between a counterexample and non-example, or generation of non existing counterexamples.

There is a consensus among mathematics educators and researches that students' understanding of a concept is influenced by their overall experiences with the concept. This implies to the concept of counterexample as well. The purpose of this study was to present students with a kind of learning environment and experiences which facilitate the generation and use of counterexamples in order to reflect on learning processes the students go through when they engage in activities that prescribe the use of counterexamples.

For the purposes of this study a teaching unit that addresses the students' difficulties with counterexamples was especially designed (in two parallel versions adjusted for both high and low level students). The unit was implemented in two classes: high and low level students in a high school.

The findings suggest that engaging in different kinds of activities that emphasize various aspects of counterexamples, helped students to improve their understanding of a concept of counterexample and its use. The same effect was observed in both research groups, regardless of the mathematical level or age of the students.

The analysis of students' responses revealed that students:

- Came to recognize counterexamples as a legitimate kind of proof (disproof) that can stand by its own.
- Became aware of the domain of validity of mathematical universal statements.
- Increased caution in overgeneralization of conjectures.

In addition, students in both research groups improved their content knowledge, reasoning and communication skills.

The research findings include also an identification of testing patterns in evaluation of mathematical universal statements by the research participants. Those patterns reveal the different strategies that students use when they are faced with a need to determine whether a given statement is true or false.

The main contribution of this study is the development of a learning environment that focus on counterexample use and generation; its implementation in two classes, at two different levels of mathematics learning; providing a detailed description of learning processes (of students who participated in a study) associated with counterexamples.

The importance of this study lies in its being based on real classroom evidence and grounded in a research field. The research findings show that there are ways to incorporate in, school context, powerful activities that have a potential to create learning situations in witch students can develop their understanding of mathematical concepts and improve their reasoning skills through dealing with counterexamples.

In addition, the learning activities from the teaching unit can serve as learning material that addresses students' difficulties in generation and use of counterexamples. Also, they can serve as a prototype for development of similar activities for other mathematical topics and for other students' population.