Using non-standard student solutions to probe what it means to solve linear equations in school

Orly Buchbinder, Daniel Chazan

Theoretical Framing

> Why do math teachers in general, and algebra teachers in particular, always insist that students show the steps leading to a solution (their work) when solving problems on paper or on the board? Why is it that knowing that the solution of the equation 20x+5=5x+65 is 4, and just writing x=4, not sufficient?

Attempts to theorize about the rationality that undergirds the practice of teaching (in the ways that Bourdieu, 1990 theorizes about practices more broadly) suggest that such matters have to do with tacit expectations that frame social interaction (Cook & Brown, 1999; Goffman, 1997). Why teachers have certain expectations of classroom interaction can be described using the notion of a norm, though this sense of norm is more tacit than descriptions in mathematics education of teachers consciously shaping classroom norms (e.g., Yackel & Cobb, 1996).

Instructional situations (blinded) each describe a set of potential classroom interactions in which particular mathematics is being taught and that mathematics influences what is to be expected of both teachers and their students. An instructional situation can be understood as a system of norms adapted to regulate the exchange of work on particular kinds of tasks for particular objects of knowledge (blinded). As an instructional situation, the solving of equations involves a market-like interaction (Bourdieu, 1998) in which students exchange their mathematical work for their teacher's verification that they have learned how to solve equations.

To make the teacher's task of assessing students' mathematical work manageable, over time, pedagogical aids have been developed for different instructional situations. One such aid in the solving equations we will call *the canonical method* for solving linear equations; it involves first adding or subtracting to isolate the variable term and then multiplying or dividing to create an equation where the variable has a coefficient of 1 and the solution can be read from the equation (see Figure 1). Thus, for students to demonstrate that they have learned to solve equations, they must show that they have performed the steps that together constitute one enactment of this method. For example, when one solves 20x+5=5x+65, one must write something like:

$$\begin{gathered} 20x+5=5x+65 \\ 15x=60 \\ x=4 \end{gathered}$$

Figure 1: An enactment of the canonical method for solving linear equations.

This theoretical perspective suggests how teachers might respond to student work that does not conform to the norms of this instructional situation. One might expect that teachers will assert that a student cannot simply:

- do the computation without recording any steps and indicate that x=4,
- graph both sides of the equation as functions and indicate that the x value of the intersection is 4,
- bring all the terms to the left, factor and then write down the value that will produce an output of 0, or
- divide all the coefficients by 5 and then solve, or and write:

$$20x+5=5x+65$$
$$4x+1=x+13$$
$$3x=12$$
$$x=4$$

Figure 2: A divide-first, non-standard solution.

When what it means to solve an equation is to use the canonical method, all of these options, even when they result in a correct answer, do not provide a teacher with enough support for arguing that students have learned to solve equations. And, if all such methods were allowed, assessing student work would become a much more complicated task. Our expectation is that such student work does not trade in this instructional situation. This expectation provides a resource for creating a breaching experiment in the sense of Garfinkel (1964) and Mehan and Wood (1975).

Objectives

The purpose of this study is to test whether there is empirical support for our conceptualization of the solving of equations as an instructional situation in which the students must learn the canonical method. We are interested in determining:

a. Whether teachers indicate that when teaching students to solve equations their responsibility is to see that students have learned the canonical method?
b. What teachers view as appropriate responses to non-standard student work?
c. What teachers indicate they themselves would do when faced with non-standard student work?

These questions are against the backdrop of other research that indicates that today, in the context of algebra word problems, teachers are willing to accept students' solutions that do not necessarily involve writing and solving equations (blinded). The overarching question is whether the kind of changes that seem to be occurring with respect to word problems are also occurring with the solving of equations?

Data sources

Practicing algebra teachers responded to whole-class scenarios in which a student presents a non-standard solution to a linear equation. Below, we report on one such scenario that involves the solution to equation whose coefficients have a common factor. A student departs from the canonical method for solving equations by dividing by this common factor first. This action is mathematically correct, and in the context of the particular equations used in the scenarios constitutes a step toward the solution. The student's action is only subtly different from the canonical method; however it is not applicable for all linear equations.

The data for this paper comes from two sources. The first one includes video and transcripts from a 3-hour session with one study group of four high school algebra teachers from a large school district who are mentors in a teacher education program. During this session, the participants viewed a comic strip of a classroom scenario in which the teacher, after seeing the non-standard student solution, realized that there is a common factor to the coefficients, and proposed to the class a new equation, in which there is no common factor. Then, the teachers watched an animation in which instead the teacher engages the class in exploring whether dividing first is a mathematically correct step or not.

The second source of data is a web-based, on-line survey around rich media representations of classrooms. As part of a multi-day, face-to-face experience, forty-five middle and high school teachers responded to an instrument that included, among other non-standard solution methods, a story in which a student divides first. Participants were first given a chance to respond in an open-ended manner to the question of what they would do next and why. Then, participants were asked to choose between four potential teacher responses: two of which validate the student work while the other two are less positive about the student solution and point out its limitations. The teachers, participating in the survey came from several large school districts. They have varied numbers of years of teaching experience, though all taught an Algebra 1 course over the last 4 years. All the participants in the on-line experience and the study groups were from the same state.

Methodology

Using Toulmin's (1969) model for examining the structure of the arguments in the everyday talk (and modifications suggested by Simosi, 2003), the data collected in a study group were analyzed linguistically (using categories from Martin & Rose, 2002) to determine the structure of the arguments presented in the teacher talk. In this model, claims – participant's judgments about what happened in the animated story - are made on the basis of data – actions in the animation. The claims are supported by warrants – justifications that the participants provided for their claims, mostly in the form of general assertions about teaching algebra. Sometimes, the claims also took the form of alternative stories, suggestions for what might have happened if the teacher had taken an alternative action. In particular, in analyzing this study group session, we sought to identify whether the teachers argue that the teaching of solving equations involves having students learn a canonical method and how to respond to a student who does not use this method.

The analysis of the data collected in the online surveys included linguistic analysis of the open-ended responses to identify the nature of the appraisals of the student work and teacher actions presented in the slideshows (Martin & Rose, 2002). Responses to multiple-choice items were analyzed using classical statistical techniques (Crocker & Algina, 2006).

Results

On the whole, the responses from both the survey and the study groups suggest that the canonical method is what students must learn, but teachers do not simply dismiss the divide-first student solution. They indicate that it is valuable for students to be able to apply the steps in the canonical method flexibly and notice when calculations can be simplified by doing the steps in a different order.

Both in the study group and in the on-line survey, the teachers agreed that the divide-first solution, proposed by the student, is a non-standard approach for solving linear equations referring to it, for example, as 'brilliant', 'creative' and 'outside the box'. In the on-line survey 22 out of 45 participants (49%) were coded as willing to accept a divide-first solution and engage a class in discussing this solution (Table 1). However, the majority of the participants, who said they would accept this line of reasoning in class, specified that they would emphasize that it is only applicable under limited circumstances and they would want other students to show additional approaches as well.

		Teachers' response to a divide-first solution		
Response categories		Accept 49% (n=22)	No explicit decision 51% (n=23)	Reject 0%
No additional actions		6	6	0
Additional comments or actions	Emphasize limitations	4	2	0
	Prompt for "other" solutions	10	11	0
	Request a canonical method	3	6	0

Table 1: Distribution of teachers' responses to the divide-first solution in the on-line survey. N=45.

Note that the numbers in the columns do not sum to the total, since some teachers mentioned more than one action they would do in case the divide-first solution would occur in their classroom.

A desire to have "other" solutions presented by students was also present with the participants who indicated neither a positive nor negative attitude towards a divide-first solution. It is unclear whether "other solutions" refers to ones using the canonical method or to a general acceptance of multiple solution methods for solving liner equations. Only 9 out of 45 (20%) of the participants indicated explicitly that they would insist on having a canonical solution presented in class, either alongside or instead of the divide-first solution.

In the on-line survey, we found no explicit rejection of the divide-first solution method, which we attribute to its mathematical validity, though some teachers expressed a concern that the majority of students might find such a solution method confusing. The participants in the study group shared this concern. Beside an appreciation for the creativity of the divide-first solution, they expressed a commitment to teaching the canonical method that works for all problems. The teachers shared their techniques for achieving this goal such as providing a general format for solving equations and crediting only those solution steps which are parts of this format. In addition, the study group teachers favored the alternative story in which the teacher changes the task assigned to students by proposing an equation that does not have terms that have a common factor.

Scientific significance of the study

Our findings suggest that teachers view themselves as responsible for assuring that students have learned the canonical method for solving linear equations. At the same time, in contrast to general criticisms of the lack of flexibility of secondary mathematics teachers, in this sample, there seems to be some openness to creativity and willingness to accept the divide-first solution. The results reported above suggest that teachers are willing to accept an unusual variation of the canonical method, while making explicit its limitations as a general strategy.

Results from the full corpus of data will allow for comparison to results about algebra teachers' acceptance of solutions to word problems that do not involve solving equations (blinded). While in the context of algebra word problems there seems to be some openness to accepting non-typical solutions that do not involve equations (blinded), our findings provide the basis to expect that teachers will be less willing to do so in the context of solving linear equations. This is evident in the fact that only 6 participants accepted the unusual solution method without reservation

Our data will also allow us to explore whether teachers are willing to entertain a different symbolic solution strategy, arithmetic reasoning, and solutions involving graphs. These results will allow us to understand to what degree reform-minded practices have had an influence on the solving of equations, in addition to the doing of word problems. Replication of this study in different states will allow us to determine the degree to which policy contexts in different states are influential in shaping teachers' responses to classroom scenarios.

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