HOW TO DECIDE? STUDENTS' WAYS OF DETERMINING THE VALIDITY OF MATHEMATICAL STATEMENTS

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This article presents an overview of an ongoing study on mathematical reasoning patterns of high school students. The initial findings came out of a study that examined learning processes and student understandings related to the concept of counterexample. More specifically, it examined the ways in which students understand and use counterexamples in mathematics, in the course of studying a special unit designed to foster opportunities for determining the validity of numerous mathematical statements. During the study a number of strategies, which students employed in the process of evaluating the validity of mathematical statements, were identified. These strategies involved a range of underlying cognitive processes that became the main focus of the current research.

BACKGROUND

Student learning and understanding of mathematical proof has been one of the main issues of mathematics education research (e.g. Fischbein, 1982; Hoyles, 1997; Harel, 2002; Mariotti, 2006). Fischbein (1982) addressed the tensions between students' formal and empirical approaches to proof. Hoyles (1997) found that students have difficulties with proof, which derive from their reliance on empirical findings. There is also evidence that students often tend to employ example-based reasoning. By this we refer to justifications that use examples to convince one's self or others regarding a certain assertion (Rissland, 1991; Zaslavsky & Shir, 2005). This is often similar to what Harel (2002) terms empirical proof scheme. In spite of the logical limitations of such reasoning in terms of generalization, it is a useful approach mathematicians often use to develop a 'guts feeling' regarding the validity of mathematical conjectures (Alcock, 2004).

From a logical perspective the use of counterexamples is very simple: one counterexample is sufficient to refute a false universal claim, i.e. claim of the form $\forall x, P(x)$. As such, counterexamples are considered an important tool in the development of mathematics (Balacheff, 1991; Lakatos, 1976). Polya (1973) emphasises the role of counterexamples as an integral part of problem solving strategies, while Michener, (1978) regards counterexamples as one of the basic elements of expert knowledge of mathematics.

Despite the seemingly simplicity of counterexamples, empirical studies indicate that students often posses wrong conceptions associated with counterexamples, their generation and use (Balacheff, 1989; Reid, 2002; Zaslavsky & Ron, 1998). Balacheff (1991) identified several different ways in which students treat counterexamples. For example, many students are reluctant to accept a single counterexample as sufficient proof of a fallacy. Students tend to reject or treat counterexamples as exceptions. Kaur & Sharon (1994) found that many college students limit the domain of examples they check when evaluating an algebraic statement to integers, ignoring negative numbers, fractions and zero.

Some logical aspects of the use of counterexamples appear to cause major difficulties to students (e.g., Helsabek, 1975; Dubinsky et al., 1988). Along this line, Zaslavsky & Ron (1997) observed several difficulties students encounter in generating and using counterexamples, e.g., the inability to distinguish for a given statement between an example that constitutes a counterexample and one that doesn't; or the generation of 'non existing' counterexamples.

THE STUDY

The purpose of the study was to examine and characterize underlying processes in which students engage when dealing with counterexamples, including difficulties they encounter. In particular, it aimed at identifying students' ways of evaluating the validity of mathematical statements (both valid and faulty), with a focus on the role of counterexamples in these processes.

For the purpose of the study, a teaching unit that addresses students' difficulties with counterexamples was especially designed (in two parallel versions adjusted for two different grade levels) and implemented in two classes: top level 10^{th} grade and low level 12^{th} grade. The activities drew on students' prior mathematical knowledge of algebra, geometry and calculus tackling various aspects of counterexamples. The teaching experiment lasted about two months, during which 6 various activities were interwoven throughout the regular curriculum.

The study was conducted in the form of action research (Ball, 2000), in which the researcher served both as developer of the learning environment and as the teacher implementing it. Most of the data was collected, during classroom activities, in the form of audio recordings of students' interactions as they worked in small groups and of whole-group classroom discussions. In addition, written pre and post questionnaires were used, as well as the researcher's journal with field notes and reflections. The questionnaires were used for the purpose of triangulation as an additional source of information about students' conceptions regarding counterexamples.

FINDINGS

The findings suggest that engaging in different kinds of activities that emphasize various aspects of counterexamples, helped students improve their understanding of the notion of counterexample and its use. The same effect was observed in both research groups, regardless of the mathematical level or age of the students. The analysis of students' responses revealed that students came to recognize counterexamples as legitimate tools for refuting false statements; they became more aware of the domain of validity of mathematical universal statements and of the caution needed to avoid overgeneralization of conjectures. In addition, students in both classes improved their content knowledge, as well as their reasoning and communication skills.

One of the most interesting finding was the range of strategies by which students approached the need to evaluate the validity of mathematical statements. These strategies can be described as various paths connecting sequences of points in which decisions need to be taken. Interestingly, similar strategies were observed in both high and low level students. Some recurring paths which we term patterns were identified.

As described above, the students' strategies consisted of sequences of decision making steps. Figure 1 (in the form of a flowchart) presents the various paths students took, including the most common type of reasoning employed at different stages.

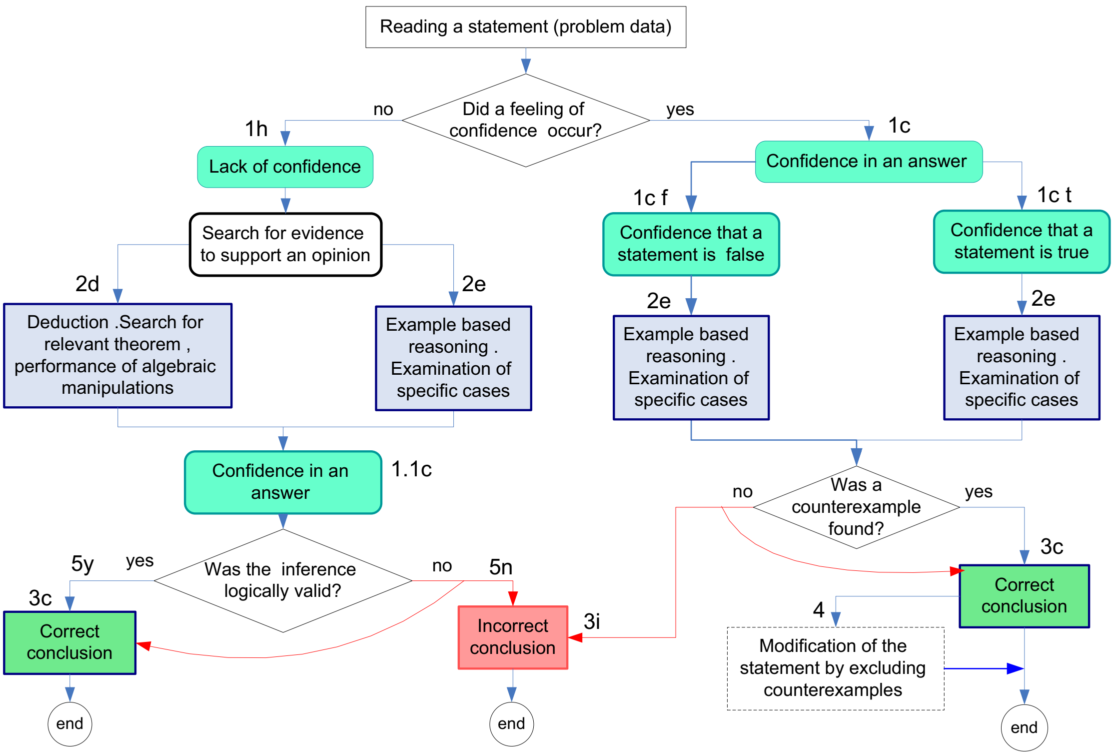

Figure 1: Students' strategies in determining the truth value of mathematical statements.

As shown in Figure 1, the first step students took for a given statement was based on their intuition and sense of confidence. If they felt confident of its truth value (see 1ct & 1cf in Figure 1), they stated their assertion and supported it by example-based reasoning, that is, by examination of specific examples (see 2e in Figure 1). The continuation of the path depended to a certain extent on the truth value of the given statement (True or False) and on the student's initial assertion (Correct or Incorrect). Thus, a student who was confident that a true statement was false, could not find any counterexample to support his assertion; or a student who asserted that a false statement was true, could have found by chance a counterexample that contradicted his assertion. In both cases, the example-based evidence had an effect on the student's final decision (see 3c & 3i in Figure 1), occasionally resulting in students shift to a different decision, sometimes accompanied by a modification of the original statement, to exclude the examples that 'didn't fit' (see 4 in Figure 1).

Students who had no 'guts feeling' in the initial stage (see 1h in Figure 1) expressed hesitation and a need to gather evidence in order to form an opinion. Some turned to example-based reasoning, while others took a deductive approach, by attempting to recall relevant theorems that can help them decide (see 2e & 2d in Figure 1). In either approach, after some time, the students reached a decision and expressed confidence about it (see 1.1c in Figure 1). Clearly, the correctness of their decision depended on the validity of their inferences (see 5y & 5n in Figure 1).

We turn to six examples that illustrate students' reasoning patterns underlying the processes of determining the truth value of a given mathematical statement, along the paths described above. We distinguish between 4 main situations: The **S**tatement may be **T**rue or **F**alse (TS / FS), and the students' initial **D**etermination could be **C**orrect or **I**ncorrect (CD / ID). Examples 1-4 illustrate 4 different cases (TS-CD, TS-ID, FS-CD, FS-ID). Examples 5 & 6 illustrate cases in which students did not come up with an initial assertion about the validity of the statement. In each of the following examples we begin with the statement the truth value of which students were asked to determine.

Example 1(TS-CD):

Statement 1: *The sum of any three odd numbers is an odd number*.

Approach: This is a valid (**T**rue) statement. Many students determined correctly from the start that it is true, without resorting explicitly to a detailed justification. In order to justify their judgment students turned to an investigation of specific examples, which we regard as example-based reasoning (Rissland, 1991; Zaslavsky & Shir, 2005). A typical response was: "Since $(-3)+1+5=3$ and $7+(-13)+(-1)=7$ the statement is always true".

This is a case where students generated a number of examples satisfying the conditions of the statement, and (not surprisingly) did not 'bump' into a counterexample. From a logical point of view, this does not constitute a proof, because theoretically there could be a counterexample that has not been found yet. However, since this statement is true, no counterexample exists.

Example 2 (TS-ID):

Statement 2: *The domains of function* $f(x)$ *and its derivative* $f'(x)$ *are not necessarily the same.*

Approach: This is a true statement, which some students determined first as false. Students' initial intrinsic feeling was that the statement is false, i.e. they wrongly asserted that the domains of any function *f(x)* and its derivative *f '(x)* must always be the same. In order to support their answer, students checked several examples, and came up with an answer, such as: "This statement is false…we tried some examples... Like $y = \sqrt{x}$ …"

In terms of students' approach, this is similar to Example 1. This answer is particularly interesting, since in the case of $f(x) = \sqrt{x}$, the domains of the function and its derivative are in fact different. This function could have served either as a counterexample to the student's initial decision, or as a proof that the statement is true. It seems like an initial intuitive feeling influenced not only students' choice of inference, but also their perception of the evidence they collected.

Example 3 (FS-CD):

In this case, students' task was to determine whether the given statement is always true, can be true in some cases or never true:

Statement 3: *In order to multiply a number by 10, you just need to write an additional "0" to its right.*

Approach: This is a false statement, which some students wrongly identified as 'always true'. In order to justify their initial assertion, they conducted a short investigation with different numbers, and came up with a counterexample (e.g., 0.4). Students accepted it as a refutation of the statement and modified their assertion. They also made an attempt to adjust its domain of validity by excluding the contradicting examples and refining the statement. Their final answer was something like: "The statement is false, since 0.4 doesn't satisfy it. But, it's true for all numbers larger than 1". It's easy to see that also the new statement proposed by the students is false.

The process of modification of a statement by excluding counterexamples and refining its domain of validity is a rational way of treating a mathematical statement (Lakatosh, 1976). But in this case students missed a crucial step. The validity of the 'new' modified statement needs to be examined. This step was ignored by students in both groups, even by students in high level class.

Ending an evaluation process without checking the validity of a new statement is logically incorrect. On several occasions students arrived at erroneous statements because they failed to check their new conjectures.

Example 4 (FS-ID):

Statement 4: *If two triangles have 2 sides and 3 angles that are equal, then the triangles are congruent.*

Approach: This is a false statement. Students (in the high level class) noticed immediately that the word "respectively" is missing and suspected that the statement is false. In order to refute this statement they initiated an explicit search for a counterexample. When they couldn't find one, students came up with the following sketch claiming it constitutes a counterexample (Figure 2).

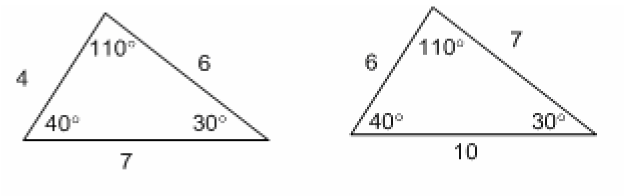

Figure 2: A non-existing 'counterexample' suggested by students

In this case, the students' intuition about the statement was correct, but they were not able to systematically construct a valid counterexample. Instead, they supported their claim by creating what they thought were two triangles that have 5 equal elements (as required) but are not congruent (although in their drawing they look as if they are congruent). However, by imposing too many conditions the resulting 'triangles' in Figure 2 do not exist, thus, cannot serve as a counterexample. At some point, during their discussion, students realized this problem, but were not able to figure out where exactly they went wrong.

<u>Example 5</u>:

Statement 5: *The function* $y = x^4 + 12x + 12$ *never gets negative values*.

Approach: Students had no initial feeling whether the statement is true or false. Thus, they expressed the need to investigate the matter further in order to gather evidence for determining the truth value of the given statement. Students first turned to consideration of special examples (e.g., negative numbers, fractions, 2, -2), that is a bottom-up approach:

Student 1: Listen to me. For this to be negative x must be a fraction.

Student 2: Why?

Student 1: If it is not a fraction, then x^4 is…more. If it's 2, then it's much bigger because…

Student 3: No. Wait. The function never gets negative values. Why? [because] if we take 2, how much is 2^4? [….] and if we take (-2)? 16. So…ok…also not good. What if we take a number smaller than 1?

Another approach was a top-down one. Students tried to retrieve an appropriate rule or theorem that would help them determine whether the given statement is false or true. An example of such an approach can be found in the following reaction:

Student 1: Can't we prove it using derivatives and all the stuff we usually do? It says here "function", and we are trying numbers. It's a function…y is above zero. So. It never gets negative values…

Student 2: Never below zero for any x.

Student 1: No. [only] if we *prove* that it's always above zero, then it's true…

As a result of both approaches, students developed confidence about the statement, i.e., they became convinced that the statement is false. Both kinds of reasoning led students to the right conclusion through logically valid inferences.

Example 6:

In this case, students' task was to determine whether the given statement is always true, can be true in some cases or never true:

Statement 6: *For any real values of* a,b,c *such that:* $a^{-1} = b^{-1} + c^{-1}$, *it follows that:* $a = b + c$.

Approach: This statement is false. Moreover, for any values of *a, b, c*, the two conditions ($a^{-1} = b^{-1} + c^{-1}$ & $a = b + c$) cannot co-exist. Similar to the case described in Example 5, here too students had no initial feeling regarding the validity of the given statement. More precisely, students were uncertain whether or not there are any values at all for *a, b, c* for which the two conditions $a^{-1} = b^{-1} + c^{-1}$ and $a = b + c$ exist. They expressed the need to investigate the matter in order to gather evidence to form an opinion. Students who chose an example-based inductive approach gave responses that were similar to: "$1^{-1} + 1^{-1} \neq 1^{-1}$ ... It's never true, because no such numbers exist. No numbers satisfy this equation."

Students who chose a deductive approach, began by performing some algebraic manipulations on the equation: $a^{-1} = b^{-1} + c^{-1}$, in order to find out whether it has any solutions. A typical response of this kind was: "$\frac{1}{a} = \frac{1}{b} + \frac{1}{c} \Rightarrow a = \frac{bc}{b+c}$ … so the statement is never true."

None of the students completed the task, although the top level students had the algebraic skills needed to do it. It seems that both example-based and deductive approaches were used by students only to gather cues regarding the truth value of the given statement. They searched for evidence that would help them form an opinion and build confidence in it. Once they were confident in their assertion, they ended the work, without noticing that their way of justification and reasoning was incomplete.

DISCUSSION

Our findings point to patterns of students' mathematical reasoning in the context of examining the truth values of mathematical statements that they have not studied beforehand. Some concur with the vast literature on proof (e.g. Harel, 2002; Balachef, 1991; Fischbein, 1987; Zaslavsky & Shir, 2005); particularly, students' reliance on intuitive evaluation of mathematical statements and their rapid use of example-based reasoning. However, there are some unique contributions of this study to our understanding of students' ways of reasoning.

Most studies, concerning students' proof practices focus on the way students prove conjectures, not on the ways they disprove them. Knuth (2002) pointed out that most classroom activities related to proof emphasise its role in validation. Students are often expected to prove results that seem obvious to them. Consequently, it is difficult for students to develop an appreciation of the need to prove. This concurs with Mariotti's view (2006) that if proof does not contribute to knowledge construction through activities that integrate a social dimension, it is likely to remain meaningless and purposeless in students' eyes.

The setting of our study relied to a large extent on the element of uncertainty as a trigger for examining the truth value of mathematical statement, in the spirit advocated by Mariotti. We provided a rich environment for fostering a genuine need for reasoning and revealing students' spontaneous approaches to justification and proof. Their search for convincing evidence was driven by their uncertainty regarding the validity of a statement, rather than by an external requirement to prove. Students had to determine the validity of mathematical statements, produce arguments to support their assertions and communicate their mathematical ideas to their peers. In addition, these arguments became a subject for whole class discussions, eliciting comparisons with arguments that are acceptable, i.e., that are already stated and shared in the mathematics community (Mariotti, 2006).

This special learning environment provided us with opportunities to identify students' natural tendencies and preferences. Thus, we identified strengths and weaknesses of students' inferences. For example, there were little rejections of counterexamples by students, contrary to the findings of Balacheff (1991). On the other hand, students tended to accept statements that they had modified without testing their validity.

We would like to offer another lens through which to examine our findings. In recent years, a number of researches in the psychology of thinking and reasoning have advocated 'dual process' theories of cognition (Evans, 2003; Kahneman, 2002; Stanovich & West, 2000). However, current theories of reasoning propose that the term 'dual process' does not suggest the existence of two distinct systems, but rather two cognitive processes that might reflect different modes of one complex system (Osman, 2003). We would like to apply a dual framework to our findings since it provides useful characteristics of students' cognitive processes, but without making strong assumptions about underlying mechanisms. The word "system" is used here as a broad term for mode or process.

The dual framework contrasts implicit cognitive processes (fast, unconscious, automatic) with explicit processes (slow, conscious, and controlled). The labels "System 1" and "System 2" are associated with these two modes of cognitive functioning (Kahneman, 2002). The framework suggests four ways in which judgment may be made. (1) No intuitive response comes to mind, and the judgment is produced by System 2. (2) An intuitive judgment or intention is evoked and (2a) is endorsed by System 2; or (2b) serves as an anchor for adjustments that respond to other features of the situation; or (2c) is identified as incompatible with a subjectively valid rule, and blocked from overt expression (Kahneman, 2002).

The observed paths of students' reasoning concur with these ways, described by Kahneman. Examples 5 & 6 refer to the option (1), when no initial intuitive feeling regarding the truth value of a statement occurred. Determination whether the given statement is true or false was made by System 2, in other words, through explicit analytical process.

Examples 1-4 refer to option (2), when System 1 was evoked and students got an immediate feeling of confidence regarding the truth value of the statement. This feeling became a subject of further explicit analytical investigation, as part of the function of System 2. In the case of Example 3, students discovered a counterexample that served a basis for correcting their initial response. The initial assertion was overridden by System 2 while a counterexample served as an anchor for modification of the intuitive answer. This is consistent with option (2b) in the described above ways of judgment.

Examples 1, 2 & 4 refer to option (2a), meaning that System 1 came up with an initial response that was endorsed by System 2. In Example 1, this endorsement is justified and students arrived at a correct decision. In Examples 2 & 4 System 2 failed in its function of monitoring the output of System 1. Students' intuitive impression was so powerful, that they did not recognise a counterexample when they saw it (Example 2) or created a non existent counterexample, when they had a strong conviction that a statement is false (Example 4).

In all patterns described above, we witnessed the strong affect of implicit intuitive reactions that guided students' mathematical behaviour. This phenomenon has wide empirical and theoretical support (Fischbein, 1987).

Explicit analytical thinking was also present in students' reasoning which we documented. This can be seen in their: attempted [direct] search for inductive evidence; discovery or construction of counterexamples and when seeming appropriate - modification of statements. These manifestations constitute strong evidence that an analytical cognitive process is present in students' reasoning and is part of their thinking strategies.

Behaviour such as ad hoc modification of a statement and its acceptance without further testing, preliminary termination of investigation, and overgeneralization of inductive evidence, suggests that the strength of an intuitive impression can interfere with analytical cognitive processes. Intuitive cognitive processes may be directing the final judgment, sometimes ignoring the relevant cues or relevant content knowledge.

More research is needed to fully characterise students' strategies in determining a truth value of mathematical statements. Elaboration of those findings in extensive theoretical framework, like the dual process theory outlined here, can contribute to broader interpretation of research findings and better understanding of students' mathematical reasoning.

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