UNCERTAINTY: A DRIVING FORCE IN CREATING A NEED FOR PROVING

Orly Buchbinder & Orit Zaslavsky

Dept. of Education in Technology & Science

Technion – Israel Institute of Technology

In this article we explore ways in which uncertainty can promote the need for proving. By uncertainty we refer to situations in which a person (or group of people) contemplates over a certain conjecture, without a sense of certitude whether it is valid or not and why it is or is not. A group of experienced secondary teachers participated in a workshop that introduced tasks aimed at creating such uncertainty. The workshop was documented and analyzed with respect to the uncertainty that emerged as the participants interacted with the tasks and with each other. The analysis points to the interplay between the sense of uncertainty, the search for certitude regarding mathematical phenomena, and the need to prove.

BACKGROUND

The importance of proof in mathematics education as a reflection of the centrality of proof in mathematics has been widely acknowledged by the mathematics education community. At the same time students and teachers encounter various difficulties with understanding and construction of mathematical proof (e.g. Healy & Hoyles, 2000; Mariotti, 2006). Some of these difficulties are rooted in students' lack of understanding of the purpose of proof (Balachef, 1990). In Harel (2007) terms, the necessity principle is often violated with respect to proof and proving. One possible way to create instructional situations in which an intellectual need for proof arises intrinsically is to use tasks that evoke uncertainty and doubt (Zaslavsky, 2005).

The role of uncertainty is particular critical in Dynamic Geometry Environments (DGE) (Hadas, Hershkowitz & Schwarz, 2000), which in many cases "provides students with strong perceptual evidence that a certain property is true" (Mariotti, 2006, p.193). Thus, once convinced by means of empirical evidence provided by DGE, students often don't feel a need for additional validation in the form of deductive justification or proof.

As Zaslavsky (2008) asserts, for teachers to gain appreciation of the potential of such uncertainty in raising the need to prove, they should engage in tasks that evoke uncertainty for them as learners and problem solvers. Designing such tasks that are closely connected to the secondary school mathematics curriculum yet challenging for secondary teachers is not a trivial undertaking. The purpose of this study is to articulate some design principles of such tasks and document their contribution in terms of evoking teachers' appreciation of and need for proving.

THE STUDY

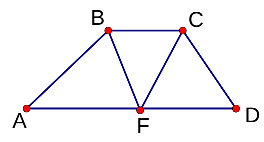
Six geometry tasks were designed for the purpose of the study (Table 1). All are related to geometry problems that appear in the commonly used geometry textbooks for secondary students in Israel. These tasks were given to a group of six experienced secondary teachers, with a sound mathematical knowledge base. Our purpose was, first, to examine the extent to which these tasks evoke uncertainty; and then to study the processes involved in dealing with them, specifically with respect to teachers' need to prove.

Each task was presented as a sequence of actions carried out by a student followed by an observation he made based on these actions. The question the teachers were asked to address was whether the observed (mathematical) phenomenon is a coincidence or not, that is, whether it holds for any case or just for some cases. Note, that there was no explicit requirement to prove any claim. The intention of the designer was to focus teachers' attention on the generality of the phenomenon, and hopefully to raise their need to form an assertion and convince themselves as well as each other by means of a proof or refutation. All six phenomena described in the tasks present observations that are not commonly addressed and have the potential of generating surprise and creating uncertainty with respect to the scope of the phenomena.

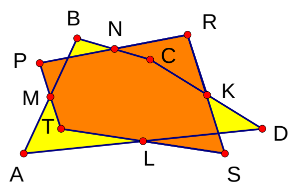
Each task presents a possible phenomenon, three of which are imperative outcomes of the described situation (tasks # 2, 4 & 6), while the other three (tasks # 1, 3 & 5) are not imperative, thus, hold just for some cases; we have termed the latter "a coincidence". The alternation of the tasks (coincidence, non-coincidence) was intended to elicit and reinforce the sense of uncertainty.

The tasks were presented to the teachers one after another in the same order that they appear here. Teachers worked in two groups or individually prior to the whole group discussion. The role of the facilitator was to navigate the discussion, summarize different points of view and pose questions.

1. A student performed the following actions: in trapezoid ABCD he drew the bisectors of two angles adjacent to the small base BC. The bisectors met at a point F on the opposite base. The student detected that the length of the base AD is equal to the sum of the two sides, AB and DC. Is this a coincidence?



2. A student performed the following actions: he took two quadrilaterals ABCD and PRST and placed one on top of the other in such way that the midpoints of their sides (K, L, M, and N) overlap. The student checked and found out that the quadrilaterals have the same areas. Is this a coincidence?



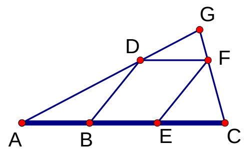
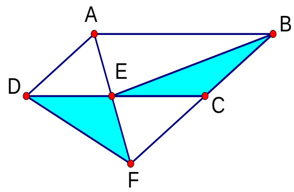
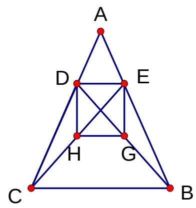
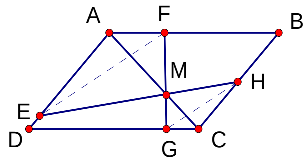
3. A student performed the following actions: he divided a segment AC into three equal parts with points B and E. Then he constructed a parallelogram BDFE. He sketched the intersection point G of rays AD and CF, and found out that they are perpendicular. Is this a coincidence?	
4. A student performed the following actions: on the side CD of a parallelogram ABCD he chose an arbitrary point E. He marked the intersection of AE and BC by F. He measured the areas of the triangles BCE and DEF and found that the areas are equal. Is this a coincidence?	
5. A student performed the following actions: in an isosceles triangle ABC (AB=AC) he drew segments of equal length: BD and CE. He marked the mid-points of these segments as G and H respectively. The student examined the quadrilateral DEGH and found out that it's a square. Is this a coincidence?	
6. A student performed the following actions: On the diagonal AC of a parallelogram ABCD he chose an arbitrary point M. He drew two segments through M: GF which intersects the sides AB and CD; and EH which intersects the sides AD and BC of the parallelogram. The student found out that the segment EF is parallel to the segment GH. Is this a coincidence?	

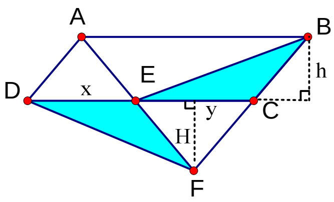
Table 1: Six tasks that were used in the study

The whole session (two hours) was videotaped and transcribed for further analysis. The data was analyzed according to two main criteria: 1. utterances that indicate uncertainty encounters; and 2. evidence for an evoked need for proving.

FINDINGS

We focus now on task # 4 (see Table 1). In this task a hypothetical student begins with a parallelogram and constructs two triangles DEF and BEC related to it, which, he claims, have the same area. The construction seemed arbitrary to the teachers, who appeared to have no initial feeling whether the result is accidental or not, and no intuition to build on. Thus, they were uncertain regarding whether this is a coincidence or not.

Teachers' initial approach was to try to prove that any such triangles have equal areas. For this they introduced parameters and expressed a relationship that would hold if triangles DEF and BEC indeed had equal areas (Figure 1).



If triangles DEF and BEC are of equal areas, then:

$$(*)\quad yh = xH \quad \text{or} \quad \frac{DE}{EC} = \frac{h}{H}$$

Figure 1: Teachers' representation of the necessary condition for triangles DEF and BEC to be equivalent (i.e., of equal area).

Teachers didn't know how to proceed further, since they couldn't think of any explanation why this property should hold. This reinforced their initial feeling of uncertainty. In Zaslavsky's (2005) terms, they manifested both lack of immediate available tools for determining how to proceed and lack of guts feeling, which triggered uncertainty due to an unknown path.

The following excerpt illustrates teachers' growing perplexity.

> Stella: [referring to (*) which she wrote on the board, see Figure 1]…Is it always true? I don't think so. Let's say, I don't have a feeling that it's always true.
>
> Debora: I started to think that it's not true at all. … I couldn't find anything certain.
>
> Facilitator: Does it mean that you couldn't show that it's false?
>
> Debora: No [I couldn't].
>
> Natalie: The real question is why should it be true?
>
> Dafna: I don't see any reason for that!

Note, that even though it was explicitly stated in the task that the student has found by measuring that the triangles have equal areas, teaches started to doubt even the possibility of that, since they were not able to either confirm or refute this claim (see Debora's remark). This indicates uncertainty of non-readily verifiable outcomes type (Zaslavsky, 2005).

At this point in the discussion teachers reached an impasse. In order to proceed further the facilitator used a DGE demonstration. This demonstration (prepared in advance in the Geometer's Sketchpad environment) presented the dynamic figure related to the task, showing the measures of areas of the triangles DEF and BEC. The facilitator used the dragging function of DGE to show that the triangles always have the same areas. The following excerpt illustrates the level of surprise that was obtained by this demonstration.

> Facilitator: Here. This measures the areas. Now I'll drag the point E….
>
> Natalie: Wow!
>
> Debora: It's true! It's true!
>
> Ronit: We should applaud.
>
> Facilitator: Can you see the values (of areas)?

> Natalie: It doesn't matter! The areas are equal! … This means that the equation we've obtained earlier (*) has to hold…. But why?

DGE demonstration helped remove the uncertainty associated with respect to the possibility that the triangles could have the same area, as Natalie stated later: "The computer convinced me". In addition, it provided an empirical basis to conjecture that all triangles constructed in such way are of equal areas. But the uncertainty towards the reasons underlying this phenomenon remained, and evoked new enthusiastic attempts to prove that the phenomenon is also imperative, in order to understand why.

This appeared to be a non-trivial task for teachers. They kept expressing their astonishment after each unsuccessful attempt to prove it, by such utterances: "Amazing!"; "It seemed so unlikely before [DGE demonstration]"; "I tried to move point E to the other side, but I still can't see it", and their urge to prove it: "It's so intriguing!", "Why is this property so imperative? ... Apparently all the triangles should have it!"; "Why should it be so?"

Finally, Dafna suggested a way to prove the equivalence of the triangles and presented her idea of the proof to the whole group.

> The area of a triangle ADF equals half of the area of the parallelogram ABCD, since the triangle's base is a side AD, and its' height is the distance between two parallel sides AD and BC. By similar arguments, the sum of the areas of triangles AED and BEC is also equal to half of the area of ABCD. Thus the areas of the triangles BEC and DEF are both equal to half of the area of the ABCD minus the area of the triangle AED (Figure 1).

Teachers were very impressed by Dafna's proof, especially since in addition to showing that the triangles can indeed have the same areas, it also provided arguments explaining why this property is imperative, that is, holds for any case.

CONCLUSION

In this article we describe a set of tasks that were designed to evoke uncertainty. We provide a closer look at one such task, examining the processes involved in coping with it by a group of secondary teachers.

The design of all six tasks turned out to be effective in eliciting a strong sense of uncertainty even for experienced mathematics teachers, with respect to the possible existence of a (mathematical) phenomenon, its scope, and the underlying reasons for its existence. Note that the uncertainty evoked by the task was so intense, that in some cases it led teachers to doubt the observation stated in the task. Even when DGE demonstrations resolved uncertainty regarding the existence and scope of the phenomenon, teachers were still left with a strong sense of uncertainty regarding why it occurs. This, in turn, was a driving force for proving, as illustrated by Dafna's comment: "You can see the overwhelming emotions that it [DGE demonstration] evoked in us: the curiosity to find out the reason at any cost".

It's important to keep in mind that the task had no explicit requirement to prove. Thus, the emergence of teachers' strong intrinsic intellectual need (Harel, 2007) to prove the conjecture implied by the task can be attributed to the uncertainty evoked by it.

Teachers have openly expressed their satisfaction and excitement that they have resolved the uncertainty and found the proof. In their own words, they have expressed a sense of gain of a "valuable experience" from dealing with uncertainty in the way they did. The added value of this activity for them was their expressed recognition of the potential of tasks evoking uncertainty in creating similar processes for their students.

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